A Novel Nuclear Model for Double Beta Decay

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Abstract

The possibility of applying the Quasiparticle Tamm-Dancoff Approximation (QTDA) to describe the nuclear double beta decay is explored. Several serious inconveniences found in the Quasiparticle Random Phase Approximation (QRPA), such as: i) the extreme sensitivity of the $2\nu\beta\beta$ decay amplitudes $\mathcal{M}_{2\nu}$ on the residual interaction in the particle-particle channel, ii) the ambiguity in treating the intermediate states, and iii) the need for performing a second charge-conserving QRPA to describe the $\beta\beta$-decays to the excited final states, are not present in the QTDA. Also, the QTDA allows for explicit evaluation of energy distributions of the double-charge-exchange transition strengths and of their sum rules, and can be straightforwardly applied to single- and double-closed shell nuclei. As an example, the ^{48}Ca$\rightarrow$^{48}Ti decay is discussed within the $1fp$-shell in the particle-hole limit of the QTDA. The general [$(1,1)$-Padé-approximant-like] behavior of the $2\nu\beta\beta$-decay amplitude in the plain QRPA as well as within its different variations is briefly reviewed.

I. INTRODUCTION

It is a great pleasure and a great honor for me to contribute to this commemorative issue in memory of Dubravko Tadić, who was one of my closest friends and the best coworker I ever had. I miss him badly, as many people do! We cooperated closely since 1966. First, we studied the single beta (β)-decay [1, 2, 3, 4], and in recent years we were basically involved in the double beta ($\beta\beta$)-decay [5, 6, 7, 8], and the nonmesonic weak decay of hypernuclei [9, 10, 11, 12, 13]. These topics are nice examples of interrelation between Particle and Nuclear Physics. In fact, Dubravko Tadić took part in many important developments of the theory of weak interactions, as well as in the advancement of particle and nuclear physics as a whole. With the entanglement between birds and fishes in the Escher's engraving, shown in Figure 1, I want to symbolize the close cooperation I had with Dubravko. To tell the truth, I have done my first work in theoretical physics, entitled: *On the Induced Terms and Partial Conservation of the Axial Vector Current in Beta Decay* [1], under Dubravko's guidance, and the line of research in our last common work, entitled: *Nuclear Structure in Nonmesonic Weak Decay of Hypernuclei* [13], has also been suggested by Dubravko. Therefore it is not difficult to figure out who was the bird and who was the fish in our teamwork.

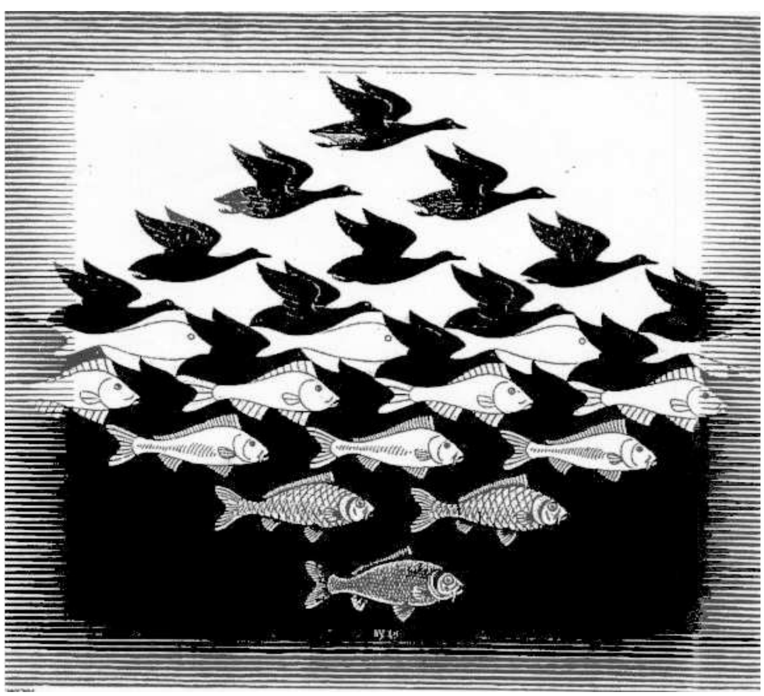

FIG. 1: Escher's engraving where the entanglement between birds and fishes pictures my joint work with Dubravko.

Among several topics on weak interactions that I have tackled with Dubravko, I will limit the present discussion to a few features of the $\beta\beta$-decay, from which we can learn about the neutrino physics, provided we know how to deal with the nuclear structure. This is a second-order weak process whose electromagnetic analogies are the atomic Raman scattering and the nuclear $\gamma\gamma$-decay [14].

In nature there are about 50 nuclear systems where the single β-decay is energetically forbidden, and therefore the $\beta\beta$-decay turns out to be the only possible mode of disintegration. It is the nuclear pairing force which causes such an "anomaly", by making the mass of the odd-odd isobar, $(N-1, Z+1)$, to be greater than the masses of its even-even neighbors, (N, Z) and $(N, -2, Z+2)$.

The modes by which the $\beta\beta$-decay can take place are connected with the neutrino (ν)-antineutrino ($\tilde{\nu}$) distinction. If ν and $\tilde{\nu}$ are defined by the transitions:

$$\begin{aligned} & n \rightarrow p + e^- + \tilde{\nu} \\ & \nu + n \rightarrow p + e^-, \end{aligned} \tag{1.1}$$

the two-neutrino double beta ($2\nu\beta\beta$) decay $(N, Z) \xrightarrow{\beta\beta^-} (N-2, Z+2)$ can occur by two successive β decays:

$$\begin{aligned} (N, Z) & \xrightarrow{\beta^-} (N-1, Z+1) + e^- + \tilde{\nu} \\ & \xrightarrow{\beta^-} (N-2, Z+2) + 2e^- + 2\tilde{\nu}, \end{aligned} \tag{1.2}$$

passing through the intermediate virtual states of the $(N-1, Z+1)$ nucleus.

However, neutrino is the only fermion that lacks a conserved additive quantum number to differentiate between ν and $\tilde{\nu}$. Thus, it is possible for the neutrino to be a Majorana particle ($\tilde{\nu} = \nu$), *i.e.* equal to its own antiparticle à la π^0 [1]. In this case the neutrinoless $\beta\beta$ ($0\nu\beta\beta$) decay is also allowed:

$$\begin{aligned} (N, Z) & \xrightarrow{\beta^-} (N-1, Z+1) + e^- + \tilde{\nu} \equiv (N-1, Z+1) + e^- + \nu, \\ & \xrightarrow{\beta^-} (N-2, Z+2) + 2e^-. \end{aligned} \tag{1.3}$$

In absence of helicity suppression (as would be natural before the parity violation has been observed) this 0ν mode is favoured over the 2ν mode by a phase-space factor of $10^7 - 10^9$: $T_{0\nu} \sim (10^{13} - 10^{15})$ yr, while $T_{2\nu} \sim (10^{20} - 10^{24})$ yr.

[1] A Dirac particle can be viewed as a combination of two Majorana particles with equal masses and opposite CP properties, in which case their contributions to the $0\nu\beta\beta$ decay cancel.

By the early 1950's several searches for the $\beta\beta$ decay were performed, inferring that $T_{2\nu+0\nu} \gtrsim 10^{17}$ yr. This pointed towards $\tilde{\nu} \neq \nu$, and prompted the introduction of the lepton number L to distinguish ν from $\tilde{\nu}$: $L = +1$ was attributed to e^- and ν and $L = -1$ to e^+ and $\tilde{\nu}$. The assumption on conserving the additive lepton number allows the $2\nu\beta\beta$ decay but forbids the $0\nu\beta\beta$ one, for which $\Delta L = 2$.

But in 1957, with the discovery of parity non-conservation in weak interactions, the question of the Majorana/Dirac character for the neutrino is raised again. That is, the neutrino was found to be left-handed (LH) and the antineutrino right-handed (RH), and in place of (1.2) one now has:

$$\begin{aligned} n &\rightarrow p + e^- + \nu_{RH}, \\ \nu_{LH} + n &\rightarrow p + e^-. \end{aligned} \tag{1.4}$$

Consequently the second process in (1.3) is forbidden because the right-handed neutrino, emitted in the first step, has the wrong helicity to be reabsorbed in the second step. For massless neutrinos, as was believed they were, there is no mixture of ν_{LH} in ν_{RH}, and the $0\nu\beta\beta$-decay cannot go through, regardless of Dirac or Majorana nature of the neutrino. This event discouraged experimental work for a long time.

However, with the development of modern gauge theories the state of affairs began to change, and over the past decade the interest in the $\beta\beta$-decay sprang up again. There are many reasons for that. The most important one is the fact that the $0\nu\beta\beta$-decay plays a decisive role in shaping the ultimate theory in any new physics beyond the standard $SU(2)_L \times U(1)$ gauge model of electroweak interactions. Moreover, no solid theoretical principle prevents neutrinos from having mass, while the most attractive extensions of the standard model require neutrinos to be massive. Nor does theory predict the scale of neutrino masses any better than it can fix the masses of quarks and charged leptons.

Yet, once the neutrino becomes massive, the helicity is not a good quantum number any more. Then, if the neutrino is in addition a Majorana particle with an effective mass $\langle m_\nu \rangle$, the mixture of ν_{LH} in ν_{RH} is proportional to $\langle m_\nu \rangle / E_\nu$, and $0\nu\beta\beta$-decay is allowed [2]. This fact inspired experimental searches in many nuclei, not only for the 0ν $\beta\beta$-decay but also for the 2ν-decay, since these two modes of disintegration are related through the the nuclear

[2] For simplicity we assume that weak interactions with right-handed currents do not play an essential role in the neutrinoless decay.

structure effects. In fact, their half-lives cast in the form:

$$T_{2\nu}^{-1} = \mathcal{G}_{2\nu}\mathcal{M}_{2\nu}^2, \qquad\qquad T_{0\nu}^{-1} = \mathcal{G}_{0\nu}\mathcal{M}_{0\nu}^2\langle m_\nu\rangle^2, \tag{1.5}$$

where $\mathcal{G}'s$ are geometrical phase space factors, and $\mathcal{M}'s$ are nuclear matrix elements (NME's). $\mathcal{M}_{2\nu}$ and $\mathcal{M}_{0\nu}$ present many similar features to the extent that it can be stated that we shall not understand the $0\nu\beta\beta$-decay unless we understand the $2\nu\beta\beta$-decay.

I will limit the discussion here to the 2ν mode, but the whole presentation that follows can be straightforwardly applied to the 0ν mode. With Dubravko we developed a full formulation of the $0\nu\beta\beta$-decay, based on the Fourier-Bessel expansion of the weak Hamiltonian, expressly adapted for nuclear structure calculations [7, 8]. We have also worked together on the "charged majoron models" [5, 6], so called because the majoron carries the unbroken $U(1)$ charge of lepton number. These models are probably the only ones that have a chance of producing the neutrinoless $\beta\beta$-decay that includes the emission of a massless majoron at a rate which could be observed in the present generation of experiments.

At present we have at our disposal beautiful data on several $2\nu\beta\beta$-decays, namely in the following nuclei: ^{48}Ca, ^{76}Ge, ^{82}Se, ^{96}Zr, ^{100}Mo, ^{116}Cd, ^{128}Te, ^{130}Te, ^{136}Xe, ^{150}Nd, ^{238}U and ^{244}Pu. The $2\nu\beta\beta$-decay turned out to be one of the slowest processes observed so far in nature and offers a unique opportunity for testing the nuclear physics techniques for half-lives $\gtrsim 10^{20}$ yr. Disappointingly, as yet, after many years of heroic efforts of many physicists, no evidence of the nonstandard $\beta\beta$-decay has appeared. A survey of experimental results is given in the review article by Ejiri [15].

All we know is that the massive neutrinos can seesaw. In fact, the major advances in neutrino massiveness have been based so far on the compelling evidence that favors the neutrino oscillations. They first came from atmospheric neutrino flux measurements at SuperKamiokande (SK) [16], and the solar neutrino shortages at SAGE, Gallex, GNO [17], Kamiokande and SK [18]. However, only recently, the solar SNO experiment [19], jointly with the reactor KamLAND (KL) experiment [20], and the first long baseline accelerator K2K experiment [21], firmly fixed the neutrino oscillations.

That is, we now know the neutrino mix, and we have initial values for their mixing matrix elements. We know the number of light active neutrino species and the differences between the squares of their masses. But we still don't know two crucial features of the neutrino physics, which are: a) the absolute mass scale, and b) whether the neutrino is a Majorana or a Dirac particle. Only the $0\nu\beta\beta$-decay can provide this information, and this fact has

motivated the undertaking of very attractive next generation experiments for many different isotopes, including ^{48}Ca, ^{76}Ge, ^{100}Mo, ^{116}Cd, ^{130}Te, ^{136}Xe, ^{150}Nd, and ^{160}Gd [22, 23].

The interest in neutrinos goes beyond the study of their intrinsic properties, and extends to a variety of topics in astro-nuclear physics, such as the understanding of the energy production in our sun, the synthesis of heavy elements during the r-process, the influence of neutrinos on the dynamics of a core-collapse supernova explosion and the cooling of a proto-neutronstar. The neutrino physics even appears in cosmological questions such as the role of neutrinos in the matter-antimatter asymmetry in the universe.

The outline of this paper is as follows: In Section II we list a few general features of the NME's, which are necessary for understanding what follows. We discuss in Section III the general behavior of the $\beta\beta$-decay amplitude within the charge-exchange QRPA, and we record the matching refinements as well. We develop in Section IV a simple nuclear model for the $\beta\beta$-decay, based on the well-known Quasiparticle Tamm-Dancoff Approximation (QTDA). In Section V we do the analysis of the $^{48}\mathrm{Ca}\xrightarrow{\beta\beta^-}{}^{48}\mathrm{Ti}$ 2ν-decay in the particle-hole limit of this model. A few final comments and remarks are pointed out in Section VI.

II. $2\nu\beta\beta$ MATRIX ELEMENT

Independently of the nuclear model used, and when only allowed transitions are considered, the $2\nu\beta\beta$ matrix element for the $|0^+_f\rangle$ final state reads

$$\mathcal{M}_{2\nu}(f) = \sum_{\lambda=0,1}(-)^\lambda\sum_\alpha\left[\frac{\langle 0^+_f||\mathcal{O}^{\beta^-}_\lambda||\lambda^+_\alpha\rangle\langle\lambda^+_\alpha||\mathcal{O}^{\beta^-}_\lambda||0^+\rangle}{\mathcal{D}_{\lambda^+_\alpha,f}}\right]\equiv\mathcal{M}^F_{2\nu}(f)+\mathcal{M}^{GT}_{2\nu}(f) \quad (2.1)$$

where the summation goes over all intermediate virtual states $|\lambda^+_\alpha\rangle$,

$$\mathcal{O}^{\beta^-}_\lambda = (2\lambda+1)^{-1/2}\sum_{pn}\langle p||\mathrm{O}_\lambda||n\rangle\left(c^\dagger_p c_{\bar{n}}\right)_\lambda, \qquad \text{with} \qquad \begin{cases}\mathrm{O}_0 = 1 & \text{for F}\\ \mathrm{O}_1=\sigma & \text{for GT}\end{cases} \quad (2.2)$$

are the Fermi (F) and Gamow-Teller (GT) operators for β^--decay, and $c^\dagger$ (c) are the particle creation (annihilation) operators. The corresponding β^+-decay operators are $\mathcal{O}^{\beta^+}_\lambda = \left(\mathcal{O}^{\beta^-}_\lambda\right)^\dagger$, and

$$\mathcal{D}_{\lambda^+_\alpha,f} = E_{\lambda^+_\alpha} - \frac{E_0+E_{0^+_f}}{2} = E_{\lambda^+_\alpha} - E_0 - \frac{E_{0^+_f}-E_0}{2}, \quad (2.3)$$

is the energy denominator. E_0 and $E_{0^+_f}$ are, respectively, the energy of the initial state $|0^+\rangle$ and of the final states $|0^+_f\rangle$.

Contributions from the first-forbidden operators, which appear in the multipole expansion of the weak Hamiltonian, as well as those from the weak-magnetism term and other second order corrections on the allowed $2\nu\beta\beta$-decay are not relevant for the present work and will not been tackled here. In recent years we have examined all of them rather thoroughly [8, 24].

In nuclear physics the isospin symmetry is conserved to a great extent, while the Wigner SU(4) symmetry is not. Because of this, the amplitude $\mathcal{M}^F_{2\nu}$ is often neglected in the literature. Nevertheless, one should keep in mind that, while the mean field strongly breaks the isospin symmetry, the residual force restores it almost fully. Therefore, although in many cases the final value of $\mathcal{M}^F_{2\nu}$ is small, it is recommendable to keep track of this NME during calculation so as to test the consistence of the nuclear model, as well as to fix its coupling constants.

The energy distributions of the transition strengths $|\langle\lambda^+_\alpha||\mathcal{O}^{\beta^\pm}_\lambda||0^+\rangle|^2$ link the single $\beta^\pm$-decays to the single charge-exchange reactions, such as (p,n), (n,p), *etc* [15, 25, 26, 27, 28, 29]. The total $\beta^\pm$ strengths

$$S^{\beta^\pm}_\lambda = (2\lambda+1)^{-1}\sum_\alpha|\langle\lambda^+_\alpha||\mathcal{O}^{\beta^\pm}_\lambda||0^+\rangle|^2, \tag{2.4}$$

can be expressed in the form

$$S^{\beta^\pm}_\lambda = \langle 0^+|\mathcal{O}^{\beta^\mp}_\lambda\cdot\mathcal{O}^{\beta^\pm}_\lambda|0^+\rangle \equiv (-)^\lambda(2\lambda+1)^{-1}\langle 0^+|[\mathcal{O}^{\beta^\mp}_\lambda\mathcal{O}^{\beta^\pm}_\lambda]_0|0^+\rangle, \tag{2.5}$$

when $|\lambda^+_\alpha\rangle$ is a complete set of excited states that can be reached by acting with $\mathcal{O}^{\beta^\mp}_\lambda$ on the initial state $|0^+\rangle$. It follows at once that

$$S^\beta_\lambda \equiv S^{\beta^-}_\lambda - S^{\beta^+}_\lambda = (-)^\lambda(2\lambda+1)^{-1}\langle 0^+|[\mathcal{O}^{\beta^+}_\lambda,\mathcal{O}^{\beta^-}_\lambda]_0|0^+\rangle = N-Z, \tag{2.6}$$

which is the well-known single-charge-exchange sum rule, also called Ikeda sum rule (ISR) [25], for both the F and the GT transitions.

Similarly, the $\beta\beta$-decays are closely related to the double-charge-exchange reactions, and to the spectral distribution of their strengths,

$$S^{\beta\beta^\pm}_\lambda(f) \;=\; (2\lambda+1)^{-1}|\sum_\alpha\langle 0^+_f||\mathcal{O}^{\beta^\pm}_\lambda||\lambda^+_\alpha\rangle\langle\lambda^+_\alpha||\mathcal{O}^{\beta^\pm}_\lambda|0^+\rangle|^2, \tag{2.7}$$

over the final states $|0^+_f\rangle$. The total strengths are defined as:

$$S^{\beta\beta^\pm}_\lambda \;=\; \sum_f S^{\beta\beta^\pm}_\lambda(f) = (2\lambda+1)^{-1}\sum_f|\langle 0^+_f|\mathcal{O}^{\beta^\pm}_\lambda\cdot\mathcal{O}^{\beta^\pm}_\lambda|0^+\rangle|^2 \tag{2.8}$$

and can be rewritten in the form

$$S_\lambda^{\beta\beta^\pm} = (2\lambda+1)^{-1}\langle 0^+|\mathcal{O}_\lambda^{\beta^\mp}\cdot\mathcal{O}_\lambda^{\beta^\mp}\ \mathcal{O}_\lambda^{\beta^\pm}\cdot\mathcal{O}_\lambda^{\beta^\pm}|0^+\rangle, \tag{2.9}$$

The double-charge-exchange sum rules (DSR) are:

$$S_\lambda^{\beta\beta} = S_\lambda^{\beta\beta^-} - S_\lambda^{\beta\beta^+} = (2\lambda+1)^{-1}\langle 0^+|[\mathcal{O}_\lambda^{\beta^+}\cdot\mathcal{O}_\lambda^{\beta^+},\mathcal{O}_\lambda^{\beta^-}\cdot\mathcal{O}_\lambda^{\beta^-}]|0^+\rangle, \tag{2.10}$$

which when evaluated give [30, 31]:

$$S_F^{\beta\beta} \equiv S_0^{\beta\beta} \ =\ 2(N-Z)(N-Z-1), \tag{2.11}$$

and

$$S_{GT}^{\beta\beta} \equiv S_1^{\beta\beta} \ =\ 2(N-Z)\left(N-Z-1+2S_1^{\beta^+}\right) - \frac{2}{3}C, \tag{2.12}$$

where C is a relatively small quantity and is given by [31, (5)].

III. CHARGE-EXCHANGE QUASIPARTICLE RANDOM PHASE APPROXIMATION AND BEYOND

The $\beta\beta$ decays occur in medium-mass nuclei that are often far from closed shells, and, as a consequence, most of the recent attempts to evaluate $\mathcal{M}_{2\nu}$ and $\mathcal{M}_{0\nu}$ rely on the neutron-proton QRPA, because this model is much simpler computationally than the shell model (SM). Note that the kind of correlations that these two methods include are not the same. The QRPA deals with a large fraction of nucleons in a large single-particle space, but within a modest configuration space. The shell model, by contrast, deals with a small fraction of nucleons in a limited single-particle space, but allows them to correlate in arbitrary ways within a large configuration space.

The charge-exchange QRPA has been first formulated, and applied to the allowed β-decay and to the collective GT resonance, by Halbleib and Sorensen in 1967 [32]. However, intensive implementations of QRPA to $\beta\beta$-decay began only about 20 years later when Vogel and Zirnbauer [33] discovered that the ground state correlations (GSC) play an essential role in suppressing the $2\nu\beta\beta$ rates. Soon afterwards, Civitarese, Faessler and Tomoda [34] arrived to the same conclusion. Almost simultaneously, Tomoda and Faessler [35], and Engel, Vogel and Zirnbauer [36] revealed a similar though smaller effect on the $0\nu\beta\beta$ decay.

When applied to the $\beta\beta$-decay the following two steps are performed within the standard QRPA:

1. Two charge-exchange QRPA equations are solved for the intermediate 1^+ states; one for the initial nucleus (N, Z) and one for the final nucleus $(N-2, Z+2)$. The first one,

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \omega_\alpha \begin{pmatrix} X \\ -Y \end{pmatrix}, \tag{3.1}$$

is evaluated in the BCS vacuum

$$|0^+\rangle = \prod_p (u_p + v_p c_p^\dagger c_{\bar{p}}^\dagger) \prod_n (u_n + v_n c_n^\dagger c_{\bar{n}}^\dagger)|\rangle, \tag{3.2}$$

where $|\rangle$ stands for the particle vacuum. The Eq. (3.1) describes simultaneously four nuclei: $(N-1, Z-1)$, $(N+1, Z-1)$, $(N-1, Z+1)$ and $(N+1, Z+1)$. The matrix elements of the operators $\mathcal{O}_1^{\beta^\pm}$ are:

$$\begin{aligned} \langle 1_\alpha^+ || \mathcal{O}_1^{\beta^-} || 0^+ \rangle &= \sum_{pn} \left[1_+^0(pn;1) X_{pn;1_\alpha^+} + 1_-^0(pn;1) Y_{pn;1_\alpha^+} \right], \\ \langle 1_\alpha^+ || \mathcal{O}_1^{\beta^+} || 0^+ \rangle &= \sum_{pn} \left[1_-^0(pn;1) X_{pn;1_\alpha^+} + 1_+^0(pn;1) Y_{pn;1_\alpha^+} \right], \end{aligned} \tag{3.3}$$

were

$$\begin{aligned} \Lambda_+^0(pn;\lambda) &= u_p v_n \langle p || \mathrm{O}_\lambda || n \rangle, \\ \Lambda_-^0(pn;\lambda) &= v_p u_n \langle p || \mathrm{O}_\lambda || n \rangle, \end{aligned} \tag{3.4}$$

are the unperturbed strengths. The ISR reads

$$S_{GT}^\beta = \frac{1}{3} \sum_\alpha \left[|\langle 1_\alpha^+ || \mathcal{O}_1^{\beta^-} || 0^+ \rangle|^2 - |\langle 1_\alpha^+ || \mathcal{O}_1^{\beta^+} || 0^+ \rangle|^2 \right] = N - Z. \tag{3.5}$$

In the same way the second QRPA does not deal with the $(N-1, Z+1)$ nucleus only, but entangles as well the isotopes $(N-1, Z+3)$, $(N-3, Z+1)$ and $(N-3, Z+3)$. The corresponding ISR is

$$\overline{S}_{GT}^\beta = \frac{1}{3} \sum_\alpha \left[|\langle \overline{1}_\alpha^+ || \mathcal{O}_1^{\beta^-} || \overline{0}^+ \rangle|^2 - |\langle \overline{1}_\alpha^+ || \mathcal{O}_1^{\beta^+} || \overline{0}^+ \rangle|^2 \right] = \overline{N} - \overline{Z} = N - Z - 4, \tag{3.6}$$

where the barred kets ($|\overline{0}^+\rangle$, $|\overline{1}_\alpha^+\rangle$) indicate that the quasiparticles are defined with respect to the final nucleus.

2. The equation (2.1) is substituted by one of the following two ansatz:

- <u>Method M1</u> (proposed by Vogel and Zirnbauer [33, 36]):

$$\mathcal{M}_{2\nu} = \frac{1}{2}\sum_{\alpha}\left[\frac{\langle 1^+_\alpha||\mathcal{O}_1^{\beta^+}||0^+\rangle\langle 1^+_\alpha||\mathcal{O}_1^{\beta^-}||0^+\rangle}{\omega_{1^+_\alpha}} + \frac{\langle \overline{1}^+_\alpha||\mathcal{O}_1^{\beta^+}||\overline{0}^+\rangle\langle \overline{1}^+_\alpha||\mathcal{O}_1^{\beta^-}||\overline{0}^+\rangle}{\omega_{\overline{1}^+_\alpha}}\right], \tag{3.7}$$

- Method M2 (introduced by Civitarese, Faessler and Tomoda [34, 35]):

$$\mathcal{M}_{2\nu} = 2\sum_{\alpha\alpha'}\frac{\langle \overline{1}^+_{\alpha'}||\mathcal{O}_1^{\beta^+}||\overline{0}^+\rangle\langle \overline{1}^+_{\alpha'}|1^+_\alpha\rangle\langle 1^+_\alpha||\mathcal{O}_1^{\beta^-}||0^+\rangle}{\omega_{1^+_\alpha}+\omega_{\overline{1}^+_{\alpha'}}}, \tag{3.8}$$

where the overlap in (3.4) is evaluated as:

$$\langle \overline{1}^+_{\alpha'}|1^+_\alpha\rangle = \sum_{pn}\left[X_{pn;1^+_\alpha}X_{pn;\overline{1}^+_{\alpha'}} - Y_{pn;1^+_\alpha}Y_{pn;\overline{1}^+_{\alpha'}}\right]. \tag{3.9}$$

Note that the last two equations for $\mathcal{M}_{2\nu}$, (3.7) and (3.8), cannot be derived mathematically, and that they are just recipes which make the applications of the QRPA to the $\beta\beta$-decay possible. Moreover, in many applications of the method M1, the energy denominator $(\omega_{1^+_\alpha}+\omega_{\overline{1}^+_{\alpha'}})/2$ in (3.4) is simply taken to be equal to $\omega_{1^+_\alpha}$. The GSC in $\langle 1^+_\alpha||\mathcal{O}_1^{\beta^-}||0^+\rangle$ and $\langle \overline{1}^+_{\alpha'}||\mathcal{O}_1^{\beta^+}||\overline{0}^+\rangle$ are also different, matching, respectively, transitions $(N,Z) \xrightarrow{\beta^+} (N+1,Z-1)$ and $(N-2,Z+2) \xrightarrow{\beta^-} (N-3,Z+3)$.

When compared to the SM, the QRPA presents the following drawbacks:

I) There is ambiguity in treating the intermediate states, and further developments must be made to match the excited states of the odd-odd nuclei based on different ground states of the initial and final even-even nuclei, as in (3.7) and (3.8).

II) The QRPA $\beta\beta$-decay amplitudes are extremely sensitive to the PN interaction in the particle-particle (pp) channel, or more precisely to the ratio between the $S=1$ and $S=0$ forces [33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. [3] Even worse, the model collapses as a whole in the physical region of t or g^{pp}.

III) The QRPA is only capable to account for the $\beta\beta$-decay into the ground state $|0^+_1\rangle$ of the final nucleus, while to deal with final excited states it is necessary to recur to another nuclear model. Usually, one solves an extra charge-conserving QRPA equation of motion for the 2^+ -excitations on the final BCS vacuum, and one assumes the final states $|2^+_1\rangle$ and $|0^+_2\rangle$ to be the one-phonon quadrupole vibration, and a member of the two-phonon quadrupole vibrational triplet, respectively [45, 46, 47, 48, 49].

[3] For a δ force this ratio is $t = v_t^{pp}/v_s^{pp}$. Usually is introduced in an ad hoc parameter, denoted by g^{pp}.

IV) In the QRPA we cannot evaluate the energy distributions of the double charge-exchange transition strengths, given by (2.7),

There have been many tries to avoid the collapse in the QRPA, so its sensitivity to the phenomenological parameters would become more realistic. A first step in this direction has been done in Ref. [37], shortly after the finding of Vogel and Zirnbauer [33], by working out a two-vacua QRPA (TVQRPA) specially tailored for the $\beta\beta$-decay. We called it so because one solves the QRPA equation of motion on the quasiparticle vacuum

$$|\tilde{0}^+\rangle = \prod_p (u_p + \bar{v}_p c_p^\dagger c_{\bar{p}}^\dagger) \prod_n (\bar{u}_n + v_n c_n^\dagger c_{\bar{n}}^\dagger)|\rangle, \tag{3.10}$$

which involves the vacua of both the initial (u, v) and final $(\bar{u}, \bar{v})$ nuclei.

The forward $(\widetilde{A})$ and the backward $(\widetilde{B})$ going matrix elements are given by

$$\begin{aligned}\widetilde{A}(pn, p'n'; J) &= (\widetilde{\epsilon_p} + \widetilde{\epsilon_n})\delta_{pn,p'n'} \\ &+ \sqrt{\rho_p\rho_n\rho_{p'}\rho_{n'}}[(u_p v_n u_{p'} v_{n'} + \bar{v}_p \bar{u}_n \bar{v}_{p'} \bar{u}_{n'})\mathrm{F}(pn, p'n'; J) \\ &+ (u_p \bar{u}_n u_{p'} \bar{u}_{n'} + \bar{v}_p v_n \bar{v}_{n'} \bar{u}_{n'})\mathrm{G}(pn, p'n'; J)],\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\widetilde{B}(pn, p'n'; J) &= \sqrt{\rho_p\rho_n\rho_{p'}\rho_{n'}}\,[(\bar{v}_p \bar{u}_n u_{p'} v_{n'} + u_p v_n \bar{v}_{p'} \bar{u}_{n'})\mathrm{F}(pn, p'n'; J) \\ &- (u_p \bar{u}_n \bar{v}_{p'} v_{n'} + \bar{v}_p v_n u_{p'} \bar{v}_{n'})\mathrm{G}(pn, p'n'; J)]\,.\end{aligned} \tag{3.12}$$

The quasiparticle energies are:

$$\widetilde{\epsilon_p} = \frac{\widetilde{\Delta}_p}{2u_p\bar{v}_p\rho_p}, \qquad \widetilde{\epsilon_n} = \frac{\widetilde{\Delta}_n}{2u_n\bar{v}_n\rho_n}, \tag{3.13}$$

with the pairing gaps given by

$$\begin{aligned}\widetilde{\Delta}_p &= -\frac{1}{2}\sum_{p'}\sqrt{\frac{2j_{p'}+1}{2j_p+1}}u_{p'}\bar{v}_{p'}\rho_{p'}G(pp, p'p'; 0), \\ \widetilde{\Delta}_n &= -\frac{1}{2}\sum_{n'}\sqrt{\frac{2j_{n'}+1}{2j_n+1}}u_{n'}\bar{v}_{n'}\rho_{n'}G(nn, n'n'; 0).\end{aligned} \tag{3.14}$$

The factors ρ_p and ρ_n are defined as $\rho_p^{-1} = u_p^2 + \bar{v}_p^2$, $\rho_n^{-1} = \bar{u}_n^2 + v_n^2$, while the remaining notations have standard meanings [29, 39].

The $\mathcal{M}_{2\nu}$ amplitude reads

$$\mathcal{M}_{2\nu} = \sum_\alpha \frac{\langle\tilde{1}_\alpha^+||\mathcal{O}_1^{\beta^+}||\tilde{0}^+\rangle\langle\tilde{1}_\alpha^+||\mathcal{O}_1^{\beta^-}||\tilde{0}^+\rangle}{\widetilde{\omega}_{1_\alpha^+}} \tag{3.15}$$

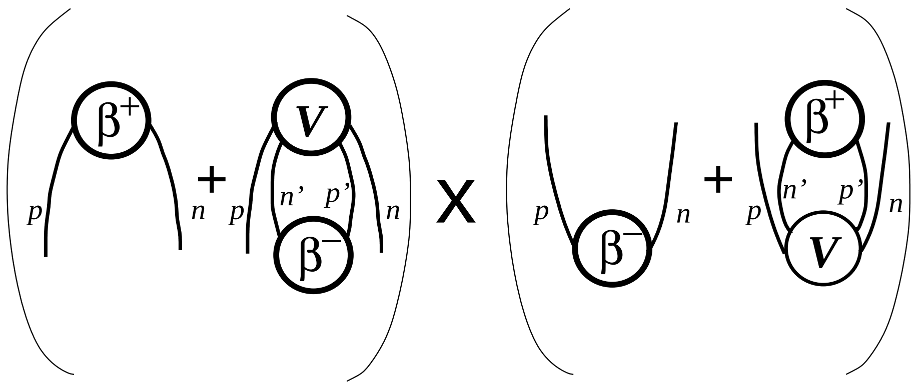

FIG. 2: Graphical representation of the numerator in (2.1) within the QRPA. The first and second terms match, respectively, to $\langle 0_1^+||\mathcal{O}_\lambda^{\beta^-}||\lambda_\alpha^+\rangle$ and $\langle \lambda_\alpha^+||\mathcal{O}_\lambda^{\beta^-}||0^+\rangle$. The residual interaction brings about the ground-state correlations in the vertex V.

where

$$\begin{aligned}
\langle \tilde{1}_\alpha^+||\mathcal{O}_1^{\beta^-}||\tilde{0}^+\rangle &= \sum_{pn}\left[\tilde{\Lambda}_+^0(pn;1)\widetilde{X}_{pn;1_\alpha^+} + \tilde{\Lambda}_-^0(pn;1)\widetilde{Y}_{pn;1_\alpha^+}\right],\\
\langle \tilde{1}_\alpha^+||\mathcal{O}_1^{\beta^+}||\tilde{0}^+\rangle &= \sum_{pn}\left[\tilde{\Lambda}_-^0(pn;1)\widetilde{X}_{pn;1_\alpha^+} + \tilde{\Lambda}_+^0(pn;1)\widetilde{Y}_{pn;1_\alpha^+}\right],
\end{aligned} \tag{3.16}$$

are, respectively, the perturbed β^- and β^+ matrix elements, and

$$\begin{aligned}
\tilde{\Lambda}_+^0(pn;\lambda) &= \sqrt{\rho_p\rho_n}u_pv_n\langle p||\mathrm{O}_\lambda||n\rangle,\\
\tilde{\Lambda}_-^0(pn;\lambda) &= \sqrt{\rho_p\rho_n}\bar{v}_p\bar{u}_n\langle p||\mathrm{O}_\lambda||n\rangle,
\end{aligned} \tag{3.17}$$

are the unperturbed ones. The ISR is now

$$\widetilde{S}_{GT}^{\beta} = \frac{1}{3}\sum_\alpha\left[|\langle\tilde{1}_\alpha^+||\mathcal{O}_1^{\beta^-}||\tilde{0}^+\rangle|^2 - |\langle\tilde{1}_\alpha^+||\mathcal{O}_1^{\beta^+}||\tilde{0}^+\rangle|^2\right] \cong N - Z - 2. \tag{3.18}$$

Now and then the gap equations are solved for the intermediate nucleus in which case the ISR gives exactly $N - Z - 2$ [6, 8, 52].

In Figure 2 we exhibit the graphical representation of different terms within the numerator in (3.13). The TVQRPA involves only the virtual $(N-1, Z+1)$-states, being the backward-going contributions for the β^- transitions the forward-going contributions for the β^+ transitions, and vice versa.

Based on a numerical comparison, we have found that the behavior of 2ν transition amplitude in the TVQRPA, with regard to the pp coupling, is quite similar to that previously noticed in the QRPA [29]. Since no uncertainties are present when intermediate states are treated in the TVQRPA, this result can, in some ways, be interpreted as a justification for the averaging procedure performed in Eqs. (3.7) and (3.8).

It could also be worth noting that in the TVQRPA we can express the amplitude $\mathcal{M}_{2\nu}$ in the form:

$$\mathcal{M}_{2\nu} = \frac{1}{2}\left(\widetilde{\Lambda}^0_+, \widetilde{\Lambda}^0_-\right)\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ -\widetilde{B} & -\widetilde{A} \end{pmatrix}^{-1}\begin{pmatrix} \widetilde{\Lambda}^0_+ \\ -\widetilde{\Lambda}^0_- \end{pmatrix}, \tag{3.19}$$

and can therefore calculate the transition probability without first solving the QRPA equation.

We found the last equation to be especially useful for discussing the $\beta\beta$-decay rates within the single-mode-model (SMM) [39, 40, 43], which deals with only one intermediate state for each J^π and is the simplest version of the QRPA. Within the SMM one can express the moment $\mathcal{M}_{2\nu}$ à la Alaga [44],

$$\mathcal{M}_{2\nu} \; = \; g^{eff}_{2\nu}\mathcal{M}^0_{2\nu}, \tag{3.20}$$

i.e. as the unperturbed BCS matrix element

$$\mathcal{M}^0_{2\nu} = u_p v_n \bar{u}_n \bar{v}_p \rho_p \rho_n \frac{\langle p||O_1||n\rangle^2}{\omega^0}, \tag{3.21}$$

multiplied by an effective charge:

$$g^{eff}_{2\nu} \; = \; \left(\frac{\omega^0}{\omega_{1^+}}\right)^2 \left[1 + \frac{G(pn,pn;1^+)}{\omega^0}\right], \tag{3.22}$$

where

$$\omega^0 = -\frac{1}{4}\left[G(pp,pp;0^+) + G(nn,nn;0^+)\right], \tag{3.23}$$

is the unperturbed pairing energy between protons and neutrons. $g^{eff}_{2\nu}$ comes from the QRPA correlations, or more precisely from the interference between the forward and backward going contributions, which add coherently in the pp channel and totally out of phase in the ph channel.

Still more, as the QRPA energy ω_{1^+} is in essence a linear function of $G(1^+)/\omega^0$, we can state that, because of the GSC, the effective QRPA charge $g^{eff}_{2\nu}$ is mainly a bilinear function

of $G(1^+)/\omega^0$, or equivalently of t [40, 42], *i.e.*

$$g_{2\nu}^{eff} \sim \frac{1-t/t_0}{1-t/t_1}, \tag{3.24}$$

That is, $\mathcal{M}_{2\nu}$ passes through zero at $t = t_0$ where $G(1^+) = -\omega^0$, and has a pole at $t = t_0$ where $\omega_{1^+} = 0$.

The fact that in many situations the SMM reproduces quite well the complete QRPA calculations made us suspect that the behavior of $\mathcal{M}_{2\nu}$ with regard to t could always follow a $(1,1)$-Padé approximant of the form (3.21). Thus, we have suggested that, independently of the nuclear Hamiltonian and/or the configuration space employed in the QRPA calculation, the 2ν amplitude should unavoidably behave as [40]

$$\mathcal{M}_{2\nu} = \mathcal{M}_{2\nu}(t=0)\frac{1-t/t_0}{1-t/t_1}. \tag{3.25}$$

At variance with the bare BCS value $\mathcal{M}^0_{2\nu}$, given by (3.20), the matrix element $\mathcal{M}_{2\nu}(t=0)$ also contains the *ph*-like correlations. We have tested the relation (3.24) only for a zero-range force [40], but as far as I know there is no QRPA calculation in the literature that could be in conflict with this result [1].

One can also guess that the breakdown in the QRPA comes from the violation of the particle-number symmetry, caused by the BCS approximation. Therefore with the hope of getting out of this inconvenience we have worked out a particle number projected QRPA (PQRPA) for the charge-exchange excitations starting from the time-dependent variational principle, [41]. However, after performing numerical calculations for the $2\nu\beta\beta$-decay in ^{76}Ge, we found that in this model the $\mathcal{M}_{2\nu}$ amplitude continues to behave roughly as in the plain QRPA. Said in another way, the number projection procedure is unable to avoid the collapse.

The variation of the QRPA that has received major attention lately is the so-called renormalized QRPA (RQRPA) [50, 51, 52, 53, 54, 55, 56, 57]. The new ingredient brought up by this model is the effect of the GSC in the QRPA equation of motion itself. The important outcome of this is that the QRPA collapse does not develop anymore in the physical region of the *pp*-strength parameter. Yet, this new procedure to incorporate the GSC tones down only slightly the strong dependence of the $2\nu\beta\beta$ transition amplitude on this parameter.

[1] When the renormalization coupling constant g^{pp} is used [34] a similar expression to the Eq. (3.22) is valid (with g^{pp}'s for t's).

On the other hand we soon found [51] that the price one has to pay in the RQRPA to avoid the collapse was the non-conservation of the ISR, given by the Eq. (2.6). This violation is about $20 - 30\%$ and we cannot get away from it. It comes from the fact that the scattering part of the GT operator, when acting on the RQRPA ground state, creates states that are not contained in the model space. These terms have recently been considered in the framework of the "Fully Renormalized QRPA" (FR-QRPA) [58], whereby the ISR was successfully restored. Yet, within the FR-QRPA the $2\nu\beta\beta$ amplitude behaves similarly as in the ordinary QRPA. Namely, in this model $\mathcal{M}_{2\nu}$ passes through zero and develops a pole for values of the pp-strength parameter which are only slightly higher than those in the QRPA model.

Let us also remember that only the self-consistent QRPA (SCQRPA) theory [59, 60, 61, 62] incorporates fully the GSC, leading simultaneously to a coupling of the single-particle field to the QRPA excitations. We performed [61] a detailed comparison of the properties of the QRPA, RQRPA and SCQRPA equations in the $O(5)$ model for the F excitations, inferring that: i) before the QRPA collapses all three approaches reproduce well correct results, ii) near the transition point only the SCQRPA values are close to the exact ones, and iii) beyond that point both the SCQRPA and RQRPA yield values different from the exact ones, but the former are somewhat better. One can suspect that in realistic cases this condition prevails and, to some extent, even for the GT transitions. Such a possibility has not been explored so far. It should also be mentioned that what some authors [63] refer to as self-consistent QRPA is just the RQRPA with the introduction of some minor changes, given by Eqs. (34), (40) and (45) in Ref. [52]. [4]

We arrive therefore to the conclusion that not one of the amendments of the QRPA, proposed so far to rescue this nuclear model, was able to change qualitatively the behavior of the amplitude $\mathcal{M}_{2\nu}$, given by (3.22), unless we agree to tolerate the violation of the ISR, which could be extremely dangerous as we have no control on how it affects the definite value of $\mathcal{M}_{2\nu}$. More, neither the RQRPA nor the SCQRPA is able to evade the other three unfavorable QRPA outcomes that we have pointed out at the beginning of this section. Thus, within this QRPA scenario, instead of introducing further improvements and variations into the QRPA equation of motion, it would perhaps be a good idea "to shuffle the cards and deal again", *i.e.* to try to work out a different quenching mechanism for the NME's. In the

[4] See also comments with regard to this in Ref. [64].

next two sections we explore that possibility.

IV. QUASIPARTICLE TAMM-DANCOFF APPROXIMATION (QTDA)

Here we sketch a simple nuclear model for evaluating the $\beta\beta$ decay rates, based on the well-known QTDA [65, 66]. It will become clear immediately that the main difference in comparison to the QRPA comes from how one describes the final $(N-2, Z+2)$ nucleus.

Same as in the QRPA, we conveniently express the total Hamiltonian as

$$H = H_p + H_n + H_{pn} + H_{pp} + H_{nn} \equiv H_0 + H_{res}, \tag{4.1}$$

where H_p and H_n are the effective proton and neutron single-quasiparticle Hamiltonians (with eigenvalues ϵ_p and ϵ_n), while H_{pn}, H_{pp}, and H_{nn} are the matching effective two-quasiparticle interaction Hamiltonians among the valence quasiparticles.

We assume both i) that the initial state is the BCS vacuum in the (N, Z) nucleus, and ii) that the intermediate and final nuclear states involved in the $\beta\beta$-decay are, respectively, two and four quasiparticle excitations on this vacuum. That is:

- initial state: $|0^+\rangle = |BCS\rangle$,
- intermediate states:

$$|\lambda^+_\alpha\rangle = \sum_{pn} X_{pn;\lambda^+_\alpha} |pn; \lambda^+\rangle, \tag{4.2}$$

  with

$$|pn; \lambda^+\rangle = [a^\dagger_p a^\dagger_n]_{\lambda^+} |BCS\rangle, \tag{4.3}$$

- final states:

$$|0^+_f\rangle = \sum_{p_1p_2n_1n_2J} Y_{p_1p_2n_1n_2J;0^+_f} |p_1p_2, n_1n_2; J\rangle, \tag{4.4}$$

  with

$$|p_1p_2, n_1n_2; J\rangle = N(p_1p_2)N(n_1n_2)\{[a^\dagger_{p_1}a^\dagger_{p_2}]_J [a^\dagger_{n_1}a^\dagger_{n_2}]_J\}_0 |BCS\rangle, \tag{4.5}$$

  and

$$N(ab) = (1 + \delta_{ab})^{-1/2}. \tag{4.6}$$

  Here $a^\dagger$ (a) is the quasiparticle creation (annihilation) operator relative to the BCS vacuo.

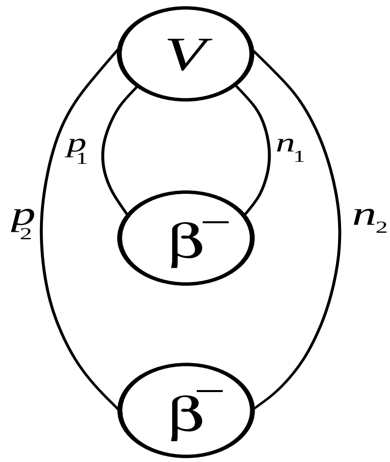

FIG. 3: Graphical representation of $\mathcal{M}_{2\nu}$ in the QTDA. The first and the second vertex match with matrix elements (4.7) and (4.8), respectively, while the third vertex represents the residual interaction in the final state.

One can read the matrix elements of H_{pn} between the intermediate states $|pn;\lambda^+\rangle$ and the final states $|p_1p_2,n_1n_2;J\rangle$, respectively from [66, (4.9) and (4.5)], and those of H_{pp} and H_{nn} between the same final states from [65, (2.11)]. We show explicit results only for the one-body matrix elements that appear in (2.1). They are:

$$\langle\lambda^+_\alpha||\mathcal{O}^{\beta^-}_\lambda||0^+\rangle \;=\; \sum_{pn}\Lambda^0_+(pn;\lambda)X_{pn;\lambda^+_\alpha}, \tag{4.7}$$

and

$$\begin{aligned}\langle 0^+_f||\mathcal{O}^{\beta^-}_\lambda||\lambda^+_\alpha\rangle \;&=\; -\sum_{pn}X_{pn;\lambda^+_\alpha}\sum_{p_1p_2n_1n_2J}Y_{\mathrm{p_1p_2n_1n_2J};0^+_f}N(p_1p_2)N(n_1n_2)\bar{P}(p_1p_2J)\bar{P}(n_1n_2J)\\ &\times\;\sqrt{2J+1}(-)^{n_1+p_2+J+\lambda}\begin{Bmatrix}p_1 & n_1 & \lambda\\ p & n & J\end{Bmatrix}\Lambda^0_+(p_1n_1;\lambda)\delta_{p_2p}\delta_{n_2n},\end{aligned} \tag{4.8}$$

where

$$\bar{P}(p_1p_2J) = 1-(-)^{p_1+p_2+J}P(p_1\leftrightarrow p_2), \tag{4.9}$$

is the well known permutation operator. The energies in the denominator (2.1) are

$$\begin{aligned}E_{\lambda^+_\alpha} \;&=\; E_0+\omega_{\lambda^+_\alpha}+\lambda_p-\lambda_n,\\ E_{0^+_f} \;&=\; E_0+\omega_{0^+_f}+2\lambda_p-2\lambda_n,\end{aligned} \tag{4.10}$$

where $\omega_{\lambda_\alpha^+}$ and $\omega_{0_f^+}$ are the eigenvalues of the Hamiltonian (4.1) for intermediate states $|\lambda_\alpha^+\rangle$ and final states $|0_f^+\rangle$, respectively, and λ_p and λ_n are the chemical potentials. Therefore

$$\mathcal{D}_{\lambda_\alpha^+,f} = \omega_{\lambda_\alpha^+} - \frac{\omega_{0_f^+}}{2}. \tag{4.11}$$

The QTDA has the correct particle-hole (shell model) limit: $v_p \to 0, v_n \to 1$, and therefore one can straightforwardly apply this model to single- and double-closed shell nuclei. In the next section we discuss an example.

V. $2\nu\beta\beta$-DECAY $^{48}Ca \to {}^{48}Ti$

There are two recent experimental results for the $2\nu\beta\beta$-decay half-life to the 0_1^+ state that nicely agree with each other. They are:

$$\begin{aligned} \text{Ref. [67]}: \qquad T_{2\nu} &= \left(4.3^{+2.4}_{-1.1}[\text{stat}] \pm 1.4[\text{syst}]\right) \times 10^{19}\ \text{yr}, \\ \text{Ref. [68]}: \qquad T_{2\nu} &= \left(4.2^{+3.3}_{-1.3}\right) \times 10^{19}\ \text{yr}, \end{aligned} \tag{5.1}$$

which, from (1.5) and the kinematical factor [42],

$$\mathcal{G}_{2\nu} = 42.3 \times 10^{-19} \left[\ \text{yr}(\text{MeV})^2\right]^{-1}, \tag{5.2}$$

yield:

$$\begin{aligned} \text{Ref. [67]}: \qquad |\mathcal{M}_{2\nu}| &= \left(0.074^{+0.040}_{-0.020}\right) [\text{MeV}]^{-1}, \\ \text{Ref. [68]}: \qquad |\mathcal{M}_{2\nu}| &= \left(0.075^{+0.015}_{-0.019}\right) [\text{MeV}]^{-1}. \end{aligned} \tag{5.3}$$

One should keep in mind that the experimental values for $|\mathcal{M}_{2\nu}|$ in (5.3) depend, through the value of $\mathcal{G}_{2\nu}$, on the value used for the effective axial-vector coupling constant g_A. In the present work we use $g_A = 1$. Clearly for the bare value, $g_A = 1.26$, the phenomenological NME's decrease by factor $(1.26)^2$.

There is also a very interesting high-resolution charge-exchange reaction experiment [69], where $\mathcal{M}_{2\nu}$ for the ground state in ^{48}Ti, was built from energy spectra of the ^{48}Ca (p, n)^{48}Sc and ^{48}Ti(d,^{2}He)^{48}Sc reactions, by converting the (p,n) and (d,^{2}He) cross sections into moments $\langle 0_1^+||\mathcal{O}_1^{\beta^-}||1_\alpha^+\rangle$ and $\langle 1_\alpha^+||\mathcal{O}_1^{\beta^-}||0^+\rangle$ which contribute in (2.1). In performing the summation over α five experimentally observed states below 5 MeV have been considered, under the assumption that all matrix elements are positive. In this way Rakers *et al.* [69] get:

$$\text{Ref. [69]}: \qquad |\mathcal{M}_{2\nu}|_{E\leq 5\,\text{MeV}} = (0.0740 \pm 0.0150)\, [\text{MeV}]^{-1}. \tag{5.4}$$

One should and that
although (5

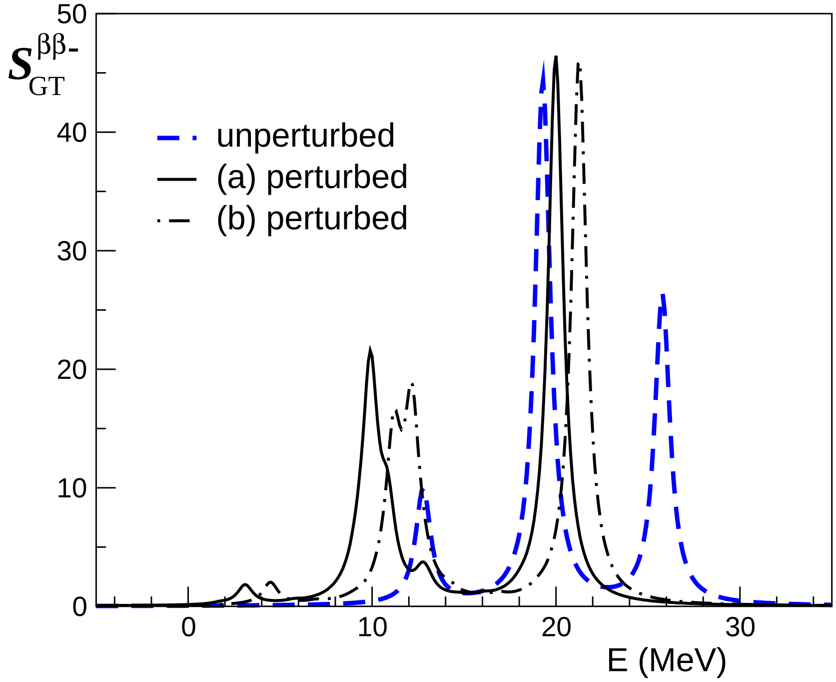

FIG. 4: Graphical representation of energy dependence of the double GT strengths $S_{GT}^{\beta\beta}$, measured from the ^{48}Ca ground state. The coupling constant values in the perturbed calculations were $v_s = 35$, $v_t = 65$ for case (a) , and $v_s = 40$, $v_t = 60$ for case (b).

Previously, jointly with Dubravko, we had already studied the $\beta\beta$ decay in ^{48}Ca, but only for the single particle state $1f_{7/2}$ [8]. Here we consider the complete pf particle-hole space. Of course, this configuration space is still strongly limited when compared to those of the SM within the pf single-particle space [70, 71, 72]. Thus, obviously, a SM yields a more realistic description of the $\beta\beta$-decay [67, 68] and of the related charge-exchange reactions [69] than the present model. Our only purpose here is to explain the way the QTDA quenching-mechanism works. Choosing suitable effective single-particle energies (SPE) is, as always, a delicate issue in nuclear structure calculations. We will take them to be:

$$
\begin{aligned}
\epsilon_{j_n} &= \epsilon^0_{j_n} + \mu_{j_n}, \\
\epsilon_{j_p} &= \epsilon^0_{j_n} + \mu_{j_p} + \Delta_C,
\end{aligned}
\tag{5.5}
$$

where $\epsilon^0_{j_n}$ are experimental SPE for ^{40}Ca, extracted from Fig. 2 in Ref. [73], namely: $\epsilon^0_{f_{7/2}} = 0$, $\epsilon^0_{f_{5/2}} = 6.5$, $\epsilon^0_{p_{3/2}} = 2.1$, and $\epsilon^0_{p_{1/2}} = 4.1$, in units of MeV. The self-energies μ_{j_n} and

μ_{j_p} account for 8 neutrons in the $f_{7/2}$ shell, and are defined in Ref. [52]. Δ_C is the Coulomb displacement energy in ^{48}Ca. With SPE chosen in this way the energy of the isobaric analog state (IAS) is always equal to Δ_C, as it should be.

We will use the δ-force

$$V = -4\pi(v_s P_s + v_t P_t)\delta(r) \tag{5.6}$$

both for evaluating the self-energies in (5.5) and for calculating the residual interaction. Besides, for the purpose of simplifying the discussion, we will assume the coupling strengths v_s and v_t to be equal for identical and for different particles.

Let us first comment the unperturbed results, *i.e.* when $v_s = v_t = 0$. In this case all double F $\beta\beta$-strength, $S_F^{\beta\beta} \equiv S_F^{\beta\beta^-} = 124$, concentrates in the lowest lying degenerate states $|f_{7/2}^2, f_{7/2}^2; J = 0, 2, 4, 6\rangle$, in parts of $S_F^{\beta\beta^-}(J) = 4(2J + 1)$. These are the double IAS's (DIAS's) and are at energy $2\Delta_C = 12.78$ MeV, measured from the ^{48}Ca ground-state, which was taken to be $E_0 = 0$. Contrarily, we found in these states only 12% of the total double GT intensity, $S_{GT}^{\beta\beta} \equiv S_{GT}^{\beta\beta^-} = 125.71$, being equal to: 2.20, 7.22, 2.64 and 3.18 for $J = 0, 2, 4$ and 6, respectively. The major part of the double GT strength concentrates at levels $|f_{7/2}^2, f_{7/2} f_{5/2}; J = 2, 4, 6\rangle$ (55%) and $|f_{7/2}^2, f_{5/2}^2; J = 0, 2, 4\rangle$ (33%), which, as seen from Figure 4 lie, respectively, at energies $2\Delta_C + \epsilon_{f_{5/2}}^0$, and $2\Delta_C + 2\epsilon_{f_{5/2}}^0$. For the lowest final 0^+ state the unperturbed denominators (4.10) are all null, which makes both NME's, $\mathcal{M}_{2\nu}^F$ and $\mathcal{M}_{2\nu}^{GT}$, to become ∞. The scene changes radically when the residual interaction is switched on. First, as we show in Figure 5, there is a great variety of physically sound values for v_s and v_t that allow the model to account for the phenomenological NME's (5.3) and (5.4). In another words, same as the QRPA, the QTDA is capable of restoring the Wigner SU(4) symmetry, quenching in this way the NME's. This is precisely the mechanism we have been searching for.

Just for the sake of illustration we use here two sets of coupling constants: (a) $v_s = 35$, $v_t = 65$, and (b) $v_s = 40$, $v_t = 60$, which reproduce reasonably well the energy levels of ^{48}Sc, shown in Figure 6, and are close to values used in our previous works [8, 29, 42, 52] within the PN ph channel ($v_s^{ph} = 27$ and $v_t^{ph} = 64$). Note first that, as $\langle f_{7/2}, f_{5/2}; 1^+|H_{pn}|f_{7/2}, f_{7/2}; 1^+\rangle \sim (v_s + v_t)$, the wave function for the GTR is the same in both cases:

$$|\mathrm{GTR}\rangle \ = \ 0.972|f_{7/2}, f_{5/2}; 1^+\rangle - 0.237|f_{7/2}, f_{7/2}; 1^+\rangle. \tag{5.7}$$

The residual interaction acts differently on double F and on GT transition strengths. In the first case, all intensity remains concentrated at the energy $2\Delta_C$, independently of the values

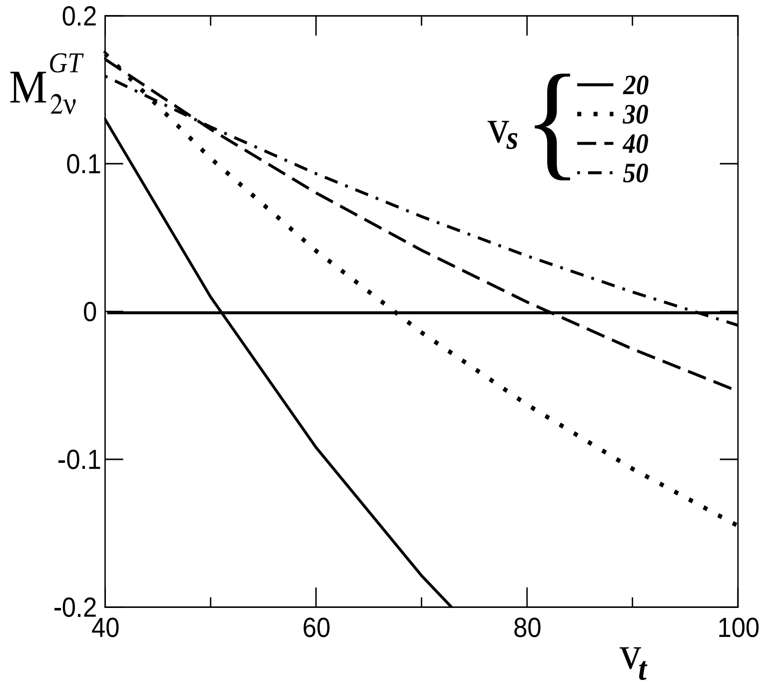

FIG. 5: Graphical representation of the $\beta\beta$ amplitudes $\mathcal{M}_{2\nu}^{GT}$ in the QTDA as a function of coupling constants v_s and v_t.

of v_s and v_t, but now there is only one DIAS. Its wave function is a coherent superposition of the states $|f_{7/2}^2, f_{7/2}^2; J = 0, 2, 4, 6\rangle$, *i.e.*

$$|\mathrm{DIAS}\rangle = \sum_J \sqrt{\frac{2J+1}{\sum_J (2J+1)}} |f_{7/2}^2, f_{7/2}^2; J\rangle. \tag{5.8}$$

In the second case the double GT resonance (DGTR), placed at an energy that is only slightly lower than $2\Delta_C + \epsilon_{f_{5/2}}^0$, carries a large part of the strength. That is: $S_{GT}^{\beta\beta^-}$ (DGTR)$= 72.87$ in case (a) and $= 72.61$ in case (b). The DGTR wave function is:

$$|\mathrm{DGTR}\rangle = \left.\begin{matrix} 0.405 \\ 0.371 \end{matrix}\right\} |f_{7/2}^2, f_{5/2}^2; 0\rangle + \left.\begin{matrix} 0.683 \\ 0.683 \end{matrix}\right\} |f_{7/2}^2, f_{5/2}^2; 2\rangle + \left.\begin{matrix} 0.505 \\ 0.533 \end{matrix}\right\} |f_{7/2}^2, f_{5/2}^2; 4\rangle + \cdots. \tag{5.9}$$

The remaining GR strength mainly comes from the states $|f_{7/2}^2, f_{7/2}f_{5/2}; J = 2, 4, 6\rangle$, and in both cases concentrates almost fully at the energy of $\sim 10 - 11$ MeV. The unperturbed and perturbed double GT strengths are confronted in Figure 4. The calculated wave function for the ground state in ^{48}Ti is

$$|0_1^+\rangle = \left.\begin{matrix} 0.922 \\ 0.922 \end{matrix}\right\} |f_{7/2}^2, f_{7/2}^2; 0\rangle - \left.\begin{matrix} 0.250 \\ 0.226 \end{matrix}\right\} |f_{7/2}^2, f_{7/2}^2; 2\rangle - \left.\begin{matrix} 0.095 \\ 0.093 \end{matrix}\right\} |f_{7/2}^2, f_{7/2}^2; 4\rangle$$

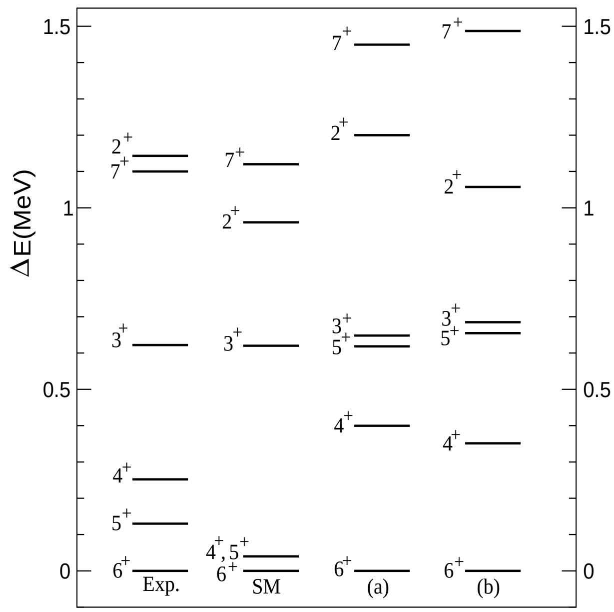

FIG. 6: Comparison between the experimental and theoretical energy levels of ^{48}Sc for the cases (a) $v_s = 35$, $v_t = 65$, and (b) $v_s = 40$, $v_t = 60$. We also display the shell-model results from Ref. [71].

$$+ \quad \left.\begin{matrix} 0.189 \\ 0.211 \end{matrix}\right\} |f^2_{7/2}, f^2_{5/2}; 0\rangle + \left.\begin{matrix} 0.119 \\ 0.138 \end{matrix}\right\} |f^2_{7/2}, p^2_{3/2}; 0\rangle + \cdots . \tag{5.10}$$

It appears at the energy $E_{0_1^+} = -4.41$ MeV in case (a) and at -3.95 MeV in case (b), which compares favorably with the measured value $Q_{\beta\beta^-} = 4.27$ MeV [67, 68, 69].

The way in which $\mathcal{M}^{GT}_{2\nu}(0^+_1)$ are constructed from individual transitions to the intermediate states $|1^+_\alpha\rangle$ is shown in Table I. We see that the GTR contributes either destructively - as in case (a), or its contribution is negligibly small - as in case (b). The first result is consistent with the SM calculation [70]. The interference effect is still more pronounced on the GT strength going to the ground state in ^{48}Ti. In fact, for all practical purposes it turns out to be null in case (a): $S^{\beta\beta^-}_{GT}(0^+_1) = (0.198 - 0.176)^2/3 = 0.000$, and extremely small in case (b): $S^{\beta\beta^-}_{GT}(0^+_1) = 0.036$. Note, however, that the quenching mainly comes from out of phase contributions among the seniority-zero and seniority-four components in the wave function (5.10), which is responsible for relatively small values of the moments $\langle 0^+_f||\mathcal{O}^{\beta^-}_1||1^+_\alpha\rangle$ shown in Table I.

The calculated GT, F and total $2\nu\beta\beta$ matrix elements are shown in Table II. One sees that the F moments are quite significant and that they cannot be neglected as is usually done. In point of fact, in several calculations, where sizable contribution to the total $\mathcal{M}_{0\nu}$

TABLE I: Decomposition of the total GT matrix elements. The denominators $\mathcal{D}_{1^+_\alpha,1}$ are given in units of MeV, and the moments $\mathcal{M}^{GT}_{2\nu}$ in units of (MeV)$^{-1}$.

α	$\langle 1^+_\alpha \| \mathcal{O}^{\beta^-}_1 \| 0^+\rangle$	$\langle 0^+_f \| \mathcal{O}^{\beta^-}_1 \| 1^+_\alpha\rangle$	$\langle 1^+_\alpha \| \mathcal{O}^{\beta^-}_1 \| 0^+\rangle \cdot \langle 0^+_f \| \mathcal{O}^{\beta^-}_1 \| 1^+_\alpha\rangle$	$\mathcal{D}_{1^+_\alpha,1}$	$\mathcal{M}^{GT}_{2\nu}$
		case (a):	$v_s = 35$, $v_t = 65$		
1	2.240	0.088	0.198	3.40	0.058
2	−4.357	0.041	−0.176	11.32	−0.016
	total				0.042
		case (b):	$v_s = 40$, $v_t = 60$		
1	2.240	0.142	0.318	4.01	0.079
2	−4.357	0.002	0.010	11.91	0.001
	total				0.080

TABLE II: Results for GT, F and total $2\nu\beta\beta$ matrix elements in units of (MeV)$^{-1}$.

case	$\mathcal{M}^{GT}_{2\nu}$	$\mathcal{M}^{F}_{2\nu}$	$\mathcal{M}_{2\nu}$
(a)	0.043	0.008	0.035
(b)	0.080	0.011	0.069

moment was found to come from the virtual 0^+ states, the F contribution to the $\mathcal{M}_{2\nu}$ moment had been omitted. This is manifestly inconsistent. [5]

Similarly, the model wave function for the excited state $|0^+_2\rangle$ in ^{48}Ti is basically composed of seniority-four basis states, *i.e.*

$$|0^+_2\rangle = \left.\begin{matrix} 0.430 \\ 0.504 \end{matrix}\right\} |f^2_{7/2}, f^2_{7/2}; 2\rangle - \left.\begin{matrix} 0.781 \\ 0.760 \end{matrix}\right\} |f^2_{7/2}, f^2_{7/2}; 4\rangle + \left.\begin{matrix} 0.373 \\ 0.307 \end{matrix}\right\} |f^2_{7/2}, f^2_{7/2}; 6\rangle + \cdots . \tag{5.11}$$

Unlike for the ground state, in case (a) appears an appreciable amount of the GT strength, $S^{\beta\beta^-}_{GT}(0^+_2) = 0.170$, in the state $|0^+_2\rangle$. Nevertheless, we will not discuss here the $\beta\beta$-decay rate

[5] A more careful study should go beyond the allowed approximation considered here and include the higher order contributions in the weak Hamiltonian as well [8, 24].

for the exited 0^+ state because the model does not reproduce satisfactorily the excitation energy; namely, we get 5.72 MeV, while the experimental value is 3.00 MeV. Finally, we note that in case (b) $S_{GT}^{\beta\beta^-}(0_2^+) = 0.038$.

VI. CONCLUDING REMARKS

The QTDA does not suffer from the inconveniences that have been listed in Section III in relation to the QRPA. More specifically, the similarities and the dissimilarities between the two models are:

1. While the QTDA contains two "β^-- like" vertices (see Fig. 3), in the QRPA we always approximate one of them by a "β^+- like" vertex (see Fig. 2). This statement is valid for all variations of the standard QRPA, such as the TVQRPA, PQRTA, RQRPA and SCQRPA.

2. The QTDA moments $\langle\lambda_\alpha^+||\mathcal{O}_\lambda^{\beta^-}||0^+\rangle$, given by (4.7), produce the F and GT resonances [28], in the same manner as in the QRPA. Furthermore, as the backward-going QRPA contributions have rather little impact on the "β^-- like" transition strength, given by the first equation in (3.5), both models yield very similar results.

3. Similarly, moments $\langle 0_1^+||\mathcal{O}_\lambda^{\beta^-}||\lambda_\alpha^+\rangle$, given by (4.8), are strongly reduced by the residual interaction, as are the QRPA moments $\langle\lambda_\alpha^+||\mathcal{O}_\lambda^{\beta^+}||0^+\rangle$, given by the second equation in (3.5), restoring in this way the isospin SU(2) and Wigner SU(4) symmetries, broken initially by the mean field. We know that in QRPA this symmetry-reestablishment takes place through the cancellation effect between the forward and the backward going contributions. In fact, in several works [8, 29, 42, 43] we have used the property of maximal restoration of the SU(4) symmetry to fix the value of the *pp* parameter t. Instead, in the QTDA the quenching comes from out of phase contributions among seniority-zero and seniority-four configurations in the wave function of the final state $|0_1^+\rangle$.

4. In QTDA, unlike in QRPA, the intermediate states $|\lambda_\alpha^+\rangle$ have a unique meaning.

5. The QTDA never collapses, and therefore the amplitude $\mathcal{M}_{2\nu}$ does not have a pole anymore, as happens in (3.24). So, one can expect that, as far as model parameters are concerned $\mathcal{M}_{2\nu}$ will behave more moderately in the QTDA than in the QRPA.

6. In the QTDA one obtains the transition amplitudes $\mathcal{M}_{2\nu}$ for the excited 0^+_f states without any additional effort. One can also easily evaluate the decays to the excited final 2^+_f states; it is sufficient to diagonalize the Hamiltonian (4.1) in the $|(p_1p_2)J_p, (n_1n_2)J_n; 2^+\rangle$ basis, and to calculate the matching transition operator (4.8).

7. In the QTDA the energy distributions of the double GT transition strengths (2.7) can be evaluated directly and, the corresponding sum rule, given by (2.12), could be occasionally violated to some extent. Contrarily, the Ikeda sum rule, given by (2.6), is always fully conserved in this model.

The remarks quoted above suggest that perhaps the QTDA might be a more "natural" and a more suitable nuclear structure framework for describing the $\beta\beta$-decay than the QRPA. In fact, one should mention again that the QRPA was originally formulated for the single $\beta^{\pm}$-decays [32], and only later adapted for the $\beta\beta$-decays via the M1 and M2 [33, 34, 35, 36] ansatz. The TVQRPA, which was mathematically tailored, specifically for the $\beta\beta$-decay, is free of these averaging procedures and has the correct BCS limit given by (3.21). Nevertheless the latest model has received rather little attention in the literature. What's more, quite recently it has been claimed that the overlap (3.9) gives rise to an additional suppression mechanism for the NME's $\mathcal{M}_{2\nu}$ and $\mathcal{M}_{0\nu}$ [74, 75]. We fully disagree with such a view.

We do not suggest that the QRPA should be substituted by the QTDA. We merely state that the use of these two nuclear models in a joint manner should very likely reduce the uncertainties in the evaluation of the NME's, which is at present one of the principal worries in the nuclear physics community, and which has engendered a great deal of activity in recent years [76, 77, 78, 79]. One cannot but highlight the Suhonen's article [78] where he argues that within the QRPA it is not possible to account simultaneously, *i.e.* with the same set of model parameters, for the simple and double β-decays.

In summary, in this work we have demonstrated that the simple version of the QTDA proposed here is able to account for the suppression of $\mathcal{M}_{2\nu}$, which was the major merit of the QRPA. Moreover, we feel that this model comprises all essential nuclear structure ingredients that are needed for describing the $\beta\beta$-decay processes. Of course, whether this is totally or only partly true has still to be tested, and it might be convenient to consider some additional refinements in the future. Their incorporation, however, should not, in principle, involve serious difficulties. For instance, the number projection procedure can

be easily implemented whenever required [40]. Regarding this issue we are convinced that, despite the present-day lack of consensus among nuclear theorists on how to derive the NME's in a direct and controlled manner, they will be able to surmount this obstacle in the near future without having to resort to extremely complicated theories. In fact, nuclear physics is not merely complicated mathematics: it requires much art to discover the most important degrees of freedom and to disentangle the underlying symmetries, for the purpose of building very imaginative models, and to skilfully manipulate the pertinent adjustable parameters. One should always keep in mind Milton's witty remark: *The very essence of truth is plainness and brightness.* Very likely, once fixed the NME's, it will be possible to answer some fundamental questions about neutrinos.

I'm very sure Dubravko would agree with all that has been said above, and this greatly encourages me to pursue the presented line of research and it is a further reason to dedicate this article to him.

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